Colman in the Irish character, with the word bocht [poor] written under it in Ogham. I doubt whether this tombstone is still to be found. My information respecting it is derived from Dr. Petrie, who furnished me with a drawing of the monument made by him several years ago. Since then many of the monuments have been broken, buried, or removed to other churchyards in the neighbourhood."

Rev. Robert Carmichael, F.T.C.D., read a Paper on Laplace's Equation and the Calculus of Quaternions.

"Early in the year 1852 it accidentally suggested itself that the celebrated Equation of Laplace's Functions, which had hitherto, for all practical purposes, baffled the powers of ordinary analysis, might possibly be solved with simplicity, and in a form admitting of useful application, by the new method of analysis discovered by Sir William Hamilton. The results of the investigation thus set on foot were published in the 'Cambridge and Dublin Mathematical Journal,' February, 1852.

"To one starting with the simpler equation,

$$D_x^2 U + D_y^2 U = 0,$$

the solution of which was known to be

$$U = \Phi(x + iy) + \Psi(x - iy),$$

where $i^2 = -1$, it seemed probable that the solution of the higher equation,

$$D_x^2 V + D_y^2 V + D_z^2 V = 0,$$

should be susceptible of deduction by the employment of two imaginaries i and j, governed by the laws

$$i^2 = -1, \quad j^2 = -1, \quad ij = -ji.$$

The integral thus deduced appeared to be

$$V = \Phi(x + iz,\ y + jz) + \Psi(x - iz,\ y - jz).$$

Unable to interpret this form, and impressed with the conviction that, to render the solution, if true, of any value, such in-

terpretation was absolutely necessary, I took the liberty of soliciting the attention of mathematicians to this point.

"Having been honoured with communications from England and France in connexion with this paper, I resumed the subject in the early part of the year 1853, and entered into correspondence with Sir William Hamilton. With his valuable assistance I hoped to be able to overcome two difficulties which seemed to lie in the way of interpretation. It appeared desirable that the form of solution should be rendered more purely symmetrical by the introduction of the third imaginary unit k, and that by the aid of the same new element the character of the solution might be rendered more purely *spatial.* In one sense this form is undoubtedly spatial. If, however, we extract from it the explicit vector-unit, we get

$$i \cos \alpha + j \sin \alpha,$$

which, as referring to an unit circle is planar, whereas it would be desirable that the explicit vector-unit should be

$$i \cos \alpha + j \cos \beta + k \cos \gamma,$$

referred to the unit sphere.

"In the month of January, 1854, Sir William Hamilton pointed out the necessity of introducing some modification in the form of the solution as stated, arising out of the non-commutative character of the terms $x + iz$, $y + jz$, and $x - iz$, $y - jz$.

"In the early part of the present year this modification was supplied by Professor Graves, but the same objections lie against the modified form:

$$V = M\Phi\,(x + iz,\ y + jz) + M\Psi\,(x - iz,\ y - jz).$$

In the first place this form is not purely symmetrical; and in the second place, its character is not purely spatial. For these reasons it seems, I would say with all due respect, improbable that any interpretation of this form can be devised which will meet the requirements of physical research.

"Now the symbolic form of Laplace's equation which was integrated was

$$(D_z - iD_x - jD_y)\,(D_z + iD_x + jD_y)\,.\,V = 0,$$

which is obviously unsymmetrical. It appears then possible, that in order to have arrived at a solution susceptible of useful application, we should not have taken this form, but one purely symmetrical, and such was pointed out nearly nine years since by Sir William Hamilton, namely,

$$(iD_x + jD_y + kD_z)^2\,.\,V = 0.$$

Now, if we confine our regard to this latter form, and substitute for the imaginary symbols real quantities a, b, c, it can readily be shown that the solution of the equation,

$$(aD_x + bD_y + cD_z)^2\,.\,V = 0$$

is

$$V = \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).\,u_0\left(e^{\frac{x}{a}},\, e^{\frac{y}{b}},\, e^{\frac{z}{c}}\right) + v_0\left(e^{\frac{x}{a}},\, e^{\frac{y}{b}},\, e^{\frac{z}{c}}\right),$$

where u_0 and v_0 are arbitrary homogenous functions, of the order zero, of the quantities respectively under them, or

$$V = \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)\Phi\left(\frac{y}{b} - \frac{z}{c},\, \frac{z}{c} - \frac{x}{a},\, \frac{x}{a} - \frac{y}{b}\right) + \Psi\left(\frac{y}{b} - \frac{z}{c},\, \frac{z}{c} - \frac{x}{a},\, \frac{x}{a} - \frac{y}{b}\right).$$

If now, in the right-hand member of these equations, we replace the real quantities a, b, c, by the imaginary symbols i, j, k, respectively, we get

$$\left(\frac{x}{i} + \frac{y}{j} + \frac{z}{k}\right).\,u_0\left(e^{\frac{x}{i}},\, e^{\frac{y}{j}},\, e^{\frac{z}{k}}\right) + v_0\left(e^{\frac{x}{i}},\, e^{\frac{y}{j}},\, e^{\frac{z}{k}}\right),$$

and

$$\left(\frac{x}{i} + \frac{y}{j} + \frac{z}{k}\right)\Phi\left(\frac{y}{j} - \frac{z}{k},\, \frac{z}{k} - \frac{x}{i},\, \frac{x}{i} - \frac{y}{j}\right) + \Psi\left(\frac{y}{j} - \frac{z}{k},\, \frac{z}{k} - \frac{x}{i},\, \frac{x}{i} - \frac{y}{j}\right),$$

and modifications of these analogous to that established so conclusively by Professor Graves, for the previous form, will give, I think, solutions of Laplace's equation, which will satisfy the conditions required.

"From the peculiar nature of the symbols it is evident that the expressions last stated may be written in the forms,

$$\rho \cdot u_0 (e^{-\xi}, e^{-\eta}, e^{-\zeta}) + v_0 (e^{-\xi}, e^{-\eta}, e^{-\zeta})$$

and

$$\rho \cdot \Phi (\eta - \zeta, \zeta - \xi, \xi - \eta) + \Psi (\eta - \zeta, \zeta - \xi, \xi - \eta),$$

where ξ, η, ζ, are the co-ordinates of the point x, y, z, in magnitude *and direction*, or, if I may presume to invent the phrase, *the components* of the points x, y, z, and ρ the vector of this same point.

"There is a peculiarity about this expression to which it may be well to solicit attention. It is known to all physicists that, in the lunar theory, and in that of the perturbed motion of pendulums, there occur equations of the form,

$$u = \frac{\theta}{2n} \cdot \sin (n\theta + a) + A \cos (n\theta + B) + \&c.,$$

implying that the value of u is not simply periodic, but admits of indefinite increase.

"Similarly, in the above expression, we observe in the first term the vector ρ outside the arbitrary function, a circumstance likely to add considerably to the interest of the physical interpretation of the solution.

"That the modified form of this expression will satisfy the requirements of physical research, appears probable from the considerations, that it must be perfectly symmetrical, that it must be spatial, and that even the notation exhibits a semi-physical character.

"The exact nature of the requisite modification I am not at present prepared to state to the Academy, but with the existence of such I am strongly impressed, and as the subject has recently attracted much attention, these remarks have been submitted in the hope of contributing to the production of a result which possesses much interest both for the mathematician and the physicist.

"It may be well to add, that the two other forms in which

the solution of Laplace's equation were presented in the paper of February, 1852,—namely,

$$V = \left\{ \begin{array}{c} \Sigma A e^{m_1 x + m_2 y} \{\cos \sqrt{(m_1^2 + m_2^2)}\, z + i_P \,.\, \sin \sqrt{(m_1^2 + m_2^2)}\, z\} \\ + \\ \Sigma B e^{m_1 x + m_2 y} \{\cos \sqrt{(m_1^2 + m_2^2)}\, z - i_P \,.\, \sin \sqrt{(m_1^2 + m_2^2)}\, z\} \end{array} \right\},$$

with its duplicate

$$V = \left\{ \begin{array}{c} \Sigma A e^{\sqrt{(m_1^2 + m_2^2)} z} \{\cos (m_1 x + m_2 y) + i_P \,.\, \sin (m_1 x + m_2 y)\} \\ + \\ \Sigma B e^{\sqrt{(m_1^2 + m_2^2)} z} \{\cos (m_1 x + m_2 y) - i_P \,.\, \sin (m_1 x + m_2 y)\} \end{array} \right\}.$$

where

$$i_P\, i = \cos a + j \sin a\,;$$

and

$$V = \left\{ \begin{array}{c} \iint \Phi (m_1, m_2)\, e^{m_1 x + m_2 y} \,.\, \cos \sqrt{(m_1^2 + m_2^2)}\, z \,.\, dm_1 dm_2 \\ + \\ \iint \Psi (m_1, m_2)\, e^{m_1 x + m_2 y} \,.\, \sin \sqrt{(m_1^2 + m_2^2)}\, z \,.\, dm_1 dm_2 \end{array} \right\},$$

with its duplicate

$$V = \left\{ \begin{array}{c} \iint \Phi (m_1, m_2) \cos (m_1 x + m_2 y)\, e^{\sqrt{(m_1^2 + m_2^2)} z} \,.\, dm_1 dm_2 \\ + \\ \iint \Psi (m_1, m_2) \sin (m_1 x + m_2 y)\, e^{\sqrt{(m_1^2 + m_2^2)} z} \,.\, dm_1 dm_2 \end{array} \right\},$$

the limits of the integrals in both cases being supposed independent of the quantities x, y, and z,—stand unaffected."

Professor Graves observed, with reference to Mr. Carmichael's paper, that he entertained great hopes that Mr. Carmichael would succeed in discovering the requisite modification of the symmetrical expression now exhibited by him to the Academy, so as to make it actually satisfy Laplace's equation. Professor Graves stated that he had pursued the same track of investigation himself; but he had abandoned it in consequence of his finding that the expression

$$\Psi \left(\frac{y}{j} - \frac{z}{k},\ \ \frac{z}{k} - \frac{x}{i},\ \ \frac{x}{i} - \frac{y}{j} \right),$$

in which Ψ denotes an arbitrary function, is not a true solution of Laplace's equation. This becomes at once apparent on trying the case in which the function just given reduces to

$$\left(\frac{y}{j} - \frac{z}{k} \right)^2, \text{ or } -(y^2 + z^2).$$

Professor Graves, D.D., read a Paper on the solution of the equation of Laplace's functions.

"It is not my design, in the present communication, to discuss the results obtained by giving particular forms to the arbitrary functions f_1 and f_2, which enter into the expression,

$$V = Mf_1\,(y + jx,\; z + kx) + Mf_2\,(y - jx,\; z - kx),$$

which I lately presented to the Academy as the complete solution of the equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0.$$

"But I propose to give some development to the general formula, in order more plainly to exhibit its nature and the mutual relation of its parts. For this purpose let us take $Mf(y + jx,\; z + kx)$, and after developing it by Taylor's Theorem, let us substitute mean products of j s, and k s for the ordinary products, according to the formulæ of p. 170. It will then assume the form

$$F_1 + jF_2 + kF_3,$$

where

$$F_1 = f - \frac{x^2}{2\,!}\left(\frac{d^2f}{dy^2} + \frac{d^2f}{dz^2}\right) + \frac{x^4}{4\,!}\left(\frac{d^4f}{dy^4} + 2\,\frac{d^4f}{dy^2dz^2} + \frac{d^4f}{dz^4}\right) - \&c.,$$

$$F_2 = x\frac{df}{dy} - \frac{x^3}{3\,!}\left(\frac{d^3f}{dy^3} + \frac{d^3f}{dydz^2}\right) + \frac{x^5}{5\,!}\left(\frac{d^5f}{dy^5} + 2\,\frac{d^5f}{dy^3dz^2} + \frac{d^5f}{dydz^4}\right) - \&c.$$

$$F_3 = x\frac{df}{dz} - \frac{x^3}{3\,!}\left(\frac{d^3f}{dy^2dz} + \frac{d^3f}{dz^3}\right) + \frac{x^5}{5\,!}\left(\frac{d^5f}{dy^4dz} + 2\frac{d^5f}{dy^2dz^3} + \frac{d^5f}{dz^5}\right) - \&c.;$$

f being used for brevity to denote $f(y, z)$. It is very easy to ascertain the law according to which the coefficients of the different powers of x are formed. In F_1 the coefficient of x^{2n} is,

$$\frac{(-1)^n}{(2n)\,!}\left\{\frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right\}^n f.$$

In F_2 the coefficient of x^{2n+1} is,

$$\frac{(-1)^n}{(2n+1)\,!}\frac{d}{dy}\left\{\frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right\}^n f;$$

and in F_3 the coefficient of x^{2n+1} is

$$\frac{(-1)^n}{(2n+1)!}\frac{d}{dz}\left\{\frac{d^2}{dy^2}+\frac{d^3}{dz^2}\right\}^n f.$$

"With respect to these expressions F_1, F_2, F_3, the following circumstances deserve notice:—

"1. They are entirely freed from imaginaries.

"2. Any one of them is a solution of Laplace's equation.

"3. They are connected together by the relations

$$\frac{dF_2}{dz}=\frac{dF_3}{dy},\quad \frac{dF_3}{dx}=\frac{dF_1}{dz},\quad \frac{dF_1}{dy}=\frac{dF_2}{dx},$$

in virtue of which, the expression

$$F_1dx+F_2dy+F_3dz$$

is an exact differential.

"4. From the fact that F_1 is a solution of Laplace's equation, it follows, that F_2 and F_3 are likewise solutions. For as $F_2=\int_0^x D_2\, F_1$.

$$(D_1^2+D_2^2+D_3^2)\, F_2=\int_0^x D_2\,(D_1^2+D_2^2+D_3^2)\, F_1=0,$$

and a similar proof applies in the case of F_3.

"5. Writing f_2 in place of $\frac{df}{dy}$ in F_2, or $\frac{df}{dz}$ in F_3, we see that

$$F_2=xf_2-\frac{x^3}{3!}\left(\frac{d^2f_2}{dy^2}+\frac{d^2f_2}{dz_2}\right)+\frac{x^5}{5!}\left(\frac{d^4f_2}{dy_4}+2\,\frac{d^4f_2}{dy^2dz^2}+\frac{d^4f_2}{dz^4}\right),\ \&c.,$$

will be a solution of Laplace's Equation, whatever function of y and z is denoted by f_2.

"6. If we add this value of F_2 to F_1, we obtain a solution involving two arbitrary functions. It is exactly in this form that Lagrange has presented the solution of Laplace's Equation in his "Mecanique Analytique," p. 520.

"7. It appears, then, that we are able to *deduce* a complete solution of Laplace's Equation from *one* of the arbitrary func-

tions Mf_1, Mf_2: and this arises from the mixture of real and imaginary quantities in them.

"8. The solution just mentioned, viz., $F_1 + F_2$, might be written in the form

$$\{1 + D_1^{-2}(D_2^2 + D_3^2)\}^{-1}(f_1 + xf_2).$$

This transformation suggests an elementary process, by means of which the solution of Laplace's function, in the form of a series arranged according to ascending powers of x, may be obtained without recourse to imaginaries. Let the equation,

$$(D_1^2 + D_2^2 + D_2^3)\,V = 0,$$

be integrated twice with respect to x; ϕ_2 and ϕ_1, two arbitrary functions of y and z, being successively introduced in the integration; it will then assume the form

$$\{1 + D_1^{-2}(D_2^2 + D_3^2)\}\,V = x\phi_2 + \phi_1.$$

Hence we shall have,

$$V = \{1 + D_1^{-2}(D_2^2 + D_3^2)\}^{-1}(x\phi_2 + \phi_1).$$

The development of the operations here indicated will actually produce a result equivalent to Lagrange's. So long ago as in February, 1848, I had suggested this mode of treating differential equations; but I had then little notion of the possibility of applying it with any success in the case of an equation so intractable as that of Laplace's coefficients."

Dr. Todd presented a rubbing made by him from an inscribed tombstone in the north transept of the church of Galway. It bears the following inscription:—

HIR · LIETH · THE · BODI · OF · ON · MORIÆRTAH · OTIERNAGH · AND · HIS · WIF · KATERINA · NIGONOHW · AND · HIS · BROTHER · TEIGE · OG · CVPERS · AN$^{\circ}$ · DNI · 1580 ·

The stone is elaborately ornamented, and bears on it also a representation of an adze and square, or rule, the emblems of the trade of coopers, to which the brothers O'Tiernagh belonged.